Extensions of some classical local moves on knot diagrams

Benjamin Audoux, Paolo Bellingeri, Jean-Baptiste Meilhan, Emmanuel Wagner

ON FORBIDDEN MOVES AND THE DELTA MOVE

BENJAMIN AUDOUX, PAOLO BELLINGERI, JEAN-BAPTISTE MEILHAN, AND EMMANUEL WAGNER

Abstract. We consider the quotient of welded knotted objects under several equivalence relations, generated respectively by self-crossing changes, Δ moves, self-virtualizations and forbidden moves. We prove that for welded objects up to forbidden moves or classical objects up to Δ moves, the notions of links and string links coincide, and that they are classified by the (virtual) linking numbers; we also prove that the Δ move is an unknotting operation for welded (long) knots. For welded knotted objects, we prove that forbidden moves imply the Δ move, the self-crossing change and the self-virtualization, and that these four local moves yield pairwise different quotients, while they collapse to only two distinct quotients in the classical case.

Introduction

The diagrammatic approach to the study of knots in 3-space enjoys a vast generalization through virtual knot theory. First developped by L. H. Kauffman in the context of knots and links [11], this theory was subsequently adapted to other kinds of *virtual knotted objects*, such as braids [18] or string links [2]. In the realm of virtual knotted objects, there are two forbidden local moves, usually called Undercrossing Commute (UC) and Overcrossing Commute (OC). When we allow OC, we obtain the class of *welded knotted objects* [8, 1], while allowing both forbidden moves yields the notion of *fused knotted objects* [10]. It is well known that braids and knots embed in their welded counterpart (see [8] and [6], respectively), while this question is still open for (string) links. This is not the case for fused objects, since all fused knots are equivalent to the unknot [9, 16]. However, the theory of fused knotted objects is not completely trivial. For instance in [5], A. Fish and E. Keyman proved that fused links that have only classical crossings are characterized by their (classical) linking numbers and, on the other hand, the unrestricted virtual braid group on n strands (the braid counterpart of fused links, see [12]) is a wreath product of $n(n-1)/2$ copies of the free group of rank 2 [3].

The first result of this paper deals with another kind of "fused knotted objects"; we will define a *fused string link* as a welded string link up to UC moves. We will show that usual string links up to forbidden moves embed in fused string links, and that fused string links are classified by their *virtual linking numbers*. As a corollary, we obtain that fused links are also classified by the virtual linking numbers, thus generalizing the above mentioned result of Fish and Keyman. This classification result can be seen as a welded analogue of the classical result of H. Murakami and Y. Nakanishi [15], which states that (string) links up to Δ moves are classified by the linking numbers. In particular, in this analogy, the UC move appears as the right welded analogue of the Δ move. The situation is thus parallel to that of our previous works [1, 2], where self-virtualization appears as the right welded analogue of the self-crossing change, in the sense that we obtained in [1] a classification result which refines Habegger-Lin's link-homotopy classification of string links.

The paper further investigates these analogies. We will prove that, for welded (string) links, the UC move implies the Δ move, the self-crossing change and the self-virtualization, and that these four local moves yield pairwise different quotients. In the classical case, however, the quotients under self-crossing changes and self-virtualizations, on the one hand, and under Δ and UC moves, on the other hand, are shown to coincide. We will also show that the Δ move implies self-crossing change for welded (string) links, thus proving that it is an unknotting operation for welded (long) knots. Note that an alternative proof of this fact was given independently and simultaneously by S. Satoh in [17]; Satoh's proof relies on a diagrammatical point of view, whereas the proof given here builds on the Gauss diagram approach.

All the results provided in this paper, combined with several known results on classical and welded string links, are summarized in the diagrams of Figure 1.

Date: October 14, 2015.

1991 *Mathematics Subject Classification.* 57M25, 57M27, 20F36.

$$\begin{tikzcd}
\mathbb{Z} \cong \mathrm{SL}_2^{\mathrm{SC}} \arrow[r, hook, two heads] \arrow[d, hook] & \mathrm{SL}_2^{\Delta} \cong \mathbb{Z} \arrow[d, hook] \arrow[dl, dash, "\sim"'] \\
\mathbb{Z} \cong \mathrm{SL}_2^{\mathrm{SV}} \arrow[r, hook, two heads] & \mathrm{SL}_2^{\mathrm{F}} \cong \mathbb{Z}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{wSL}_2^{\mathrm{SC}} \arrow[r, hook, two heads] \arrow[d] & \mathrm{wSL}_2^{\Delta} \arrow[d] \arrow[dl] \\
\mathbb{Z}^2 \cong \mathrm{wSL}_2^{\mathrm{SV}} \arrow[r, hook, two heads] & \mathrm{wSL}_2^{\mathrm{F}} \cong \mathbb{Z}^2
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Aut}_{\mathrm{C}}^0(\mathrm{RF}_n) \cong \mathrm{SL}_n^{\mathrm{SC}} \arrow[r, two heads] \arrow[d, hook] & \mathrm{SL}_n^{\Delta} \cong \mathbb{Z}^{\frac{n(n-1)}{2}} \arrow[d, hook] \\
\mathrm{Aut}_{\mathrm{C}}^0(\mathrm{RF}_n) \cong \mathrm{SL}_n^{\mathrm{SV}} \arrow[r, two heads] \arrow[ur, two heads] & \mathrm{SL}_n^{\mathrm{F}} \cong \mathbb{Z}^{\frac{n(n-1)}{2}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{wSL}_n^{\mathrm{SC}} \arrow[r, two heads] \arrow[d] & \mathrm{wSL}_n^{\Delta} \arrow[d] \\
\mathrm{Aut}_{\mathrm{C}}(\mathrm{RF}_n) \cong \mathrm{wSL}_n^{\mathrm{SV}} \arrow[r, two heads] & \mathrm{wSL}_n^{\mathrm{F}} \cong \mathbb{Z}^{n(n-1)}
\end{tikzcd}$$

FIGURE 1. Connections between quotients of SL_n and wSL_n, for $n = 2$ and $n \geq 3$.
Here, the superscripts SC, SV, Δ and F denote the quotients by the equivalence relation generated by self-crossing change, self-virtualization, Δ-move and forbidden moves, respectively.

The paper is organized as follows. All the definitions are given in Section 1.1, and we recall in Section 1.2 the known related results for classical objects. Section 2 mostly deals with welded objects. The first part is devoted to the UC quotient, and we prove there that fused (string) links are classified by the virtual linking numbers. The second part investigates the Δ move, proving that it implies the self-crossing change. It follows that it is an unknotting operation for welded (long) knots. In the last part we compare the Δ and UC quotients, proving that they coincide when restricted to classical objects.

Acknowledgments. This work began during the summer school *Mapping class groups,* 3- *and* 4-*manifolds* in Cluj in July 2015; the authors thank the organizers for the great working environment. The research of the authors was partially supported by the French ANR research project "VasKho" ANR-11-JS01-002-01.

1. MAIN OBJECTS AND THE CLASSICAL CASE

1.1. **Definitions.** We first introduce the main objects of this paper.

1.1.1. *Welded string links and welded links.*

Definition 1.1. Let $p_1, \ldots, p_n$ be $n \in \mathbb{N}^*$ points in the unit interval I. An n-component *virtual string link diagram* is an immersion L of n oriented intervals $\underset{i\in\{1,\cdots,n\}}{\sqcup} I_i$ in $I \times I$, called *strands*, such that

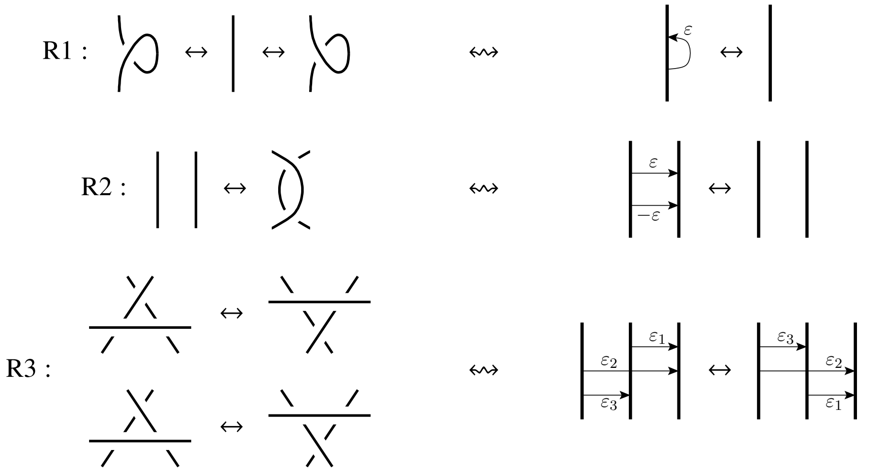
- for each $i \in \{1, \cdots, n\}$, the strand I_i has boundary $\partial I_i = \{p_i\} \times \{0, 1\}$ and is oriented from $\{p_i\} \times \{0\}$ to $\{p_i\} \times \{1\}$;
- the singular set of L is a finite number of transverse double points;
- each double point is labeled, either as a *classical crossing* or as a *virtual crossing*.

A classical crossing where the two preimages belong to the same component is called a *self-crossing*.

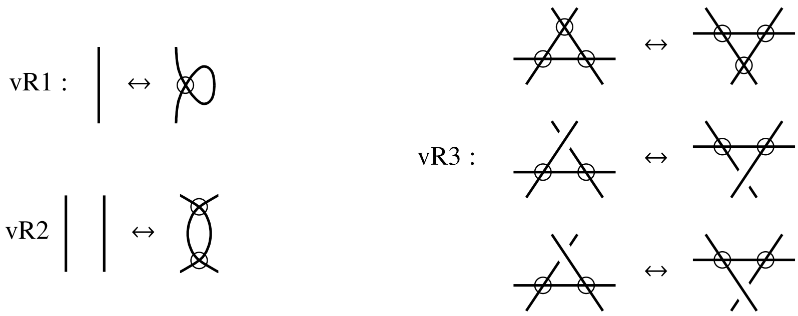
We use the usual drawing convention for the virtual and classical crossings, see e.g. Figure 2.

Up to isotopy (and reparametrization), the set of virtual string link diagrams is naturally endowed with a structure of monoid by the stacking product, where the unit element is the trivial diagram $\underset{i\in\{1,\cdots,n\}}{\cup} p_i \times I$.

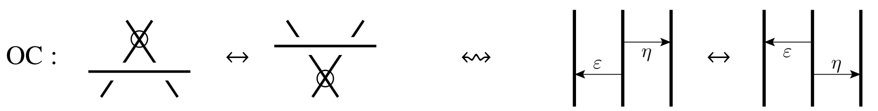
Definition 1.2. A *welded string link* is an equivalence class of virtual string link diagrams under isotopy, and the generalized (classical and virtual) Reidemeister and OC moves, represented in Figure 2.



classical Reidemeister moves, and their Gauss diagram counterparts



virtual Reidemeister moves (which have no Gauss diagram counterpart)



welded Reidemeister move, and its Gauss diagram counterpart

FIGURE 2. Generalized Reidemeister and OC moves on virtual and Gauss diagrams

We denote by wSL_n the set of welded string links; it is a monoid with composition induced by the stacking product. Elements of wSL_1 are also called welded long knots.

Similarly, one can label double points of braid and link diagrams with virtual and classical crossings, and consider the equivalence classes up to isotopy and generalized Reidemeister moves: we obtain in this way the notion of welded braids and welded links (see for instance [8]). In the following, we will denote by $w\mathcal{L}_n$ the set of n component welded links.

Definition 1.3. ([6, Section 1]) For every $i \neq j \in \{1, \dots n\}$, the *virtual linking number* $\mathrm{vlk}_{i,j}$ is the welded (string) link invariant which sends a (string) link to the sum of the signs of its classical crossings where the ith component passes over the jth component.

Note that the classical linking number $\mathrm{lk}_{i,j}$ is equal to half the sum of $\mathrm{vlk}_{i,j}$ and $\mathrm{vlk}_{j,i}$.

1.1.2. *Gauss diagrams.*

Definition 1.4. A *Gauss diagram* is a set of signed and oriented (thin) arrows between points of n ordered and oriented (thick) strands, up to isotopy of the underlying strands. Arrow endpoints are divided in two parts, *heads* and *tails*, defined by the orientation of the arrow which goes, by convention, from a tail to a head. An arrow having both ends on the same strand is called a *self-arrow*.

There is a canonical way to associate a Gauss diagram to any virtual string link diagram, so that the set of classical crossings in the virtual diagram is in one-to-one correspondence with the set of arrows in the Gauss diagram. This procedure is for example described in [1, Section 4.5] and illustrated in [1, Figure 20]. It induces a bijection between wSL_n and the set of Gauss diagrams up to the natural analogues of generalized Reidemeister and OC moves given in Figure 2. There, thick vertical lines represent portions of thick strands that can have any orientation and, outside these portions, the Gauss diagrams are identical on both sides of a given move. Distinct portions may belong to a same strands or not. Labels ε and η are either 1 or -1, but there is however a further restriction for the R3 move: the products $\varepsilon_i\delta_i$ must be equal for all $i \in \{1, 2, 3\}$, where $\delta_i = 1$ if the portion of strand disconnected from the ε_i-labeled arrow is oriented upward and $\delta_i = -1$ otherwise. Note that, up to the OC move, this restriction can be released into $\delta_2\delta_3 = \varepsilon_2\varepsilon_3$.

In the rest of the paper, we shall mostly use the Gauss diagrammatic representation. In particular, we shall consider welded invariants as defined on Gauss diagrams. For instance, it is easily checked that the virtual linking number $\mathrm{vlk}_{i,j}$ counts with signs the arrows going from strand i to strand j.

Remark 1.5. By replacing thick strands by circles in the definition of Gauss diagrams, we obtain a tool that faithfully represents welded links. Though less central in the present paper, it is more standard in the litterature, see for instance [6] or [4], and we shall use them on occasion.

Definition 1.6. A *commutation of arrows* on a Gauss diagram is the exchange of two arrow endpoints which are adjacent on a strand.

For instance, the OC move, shown in Figure 2, and the UC move, shown in Figure 3, are examples of commutations of arrows where the two endpoints are, respectively, both tails and both heads.

1.1.3. *Local moves and equivalence classes.* In Figure 3, we define several local moves on welded (string) links, namely the UC move, the Δ move, the SC move and the SV move, that we shall study in the next section.

Definition 1.7. The F–equivalence is the equivalence relation on welded string links generated by UC moves. We denote by $\mathrm{wSL}_n^{\mathrm{F}} := {}^{\mathrm{wSL}_n}\!/_{\mathrm{UC}}$ the quotient of wSL_n under F–equivalence and we call its elements n-component *fused string links*.

Definition 1.8. The Δ–equivalence is the equivalence relation on welded string links generated by Δ moves. We denote by $\mathrm{wSL}_n^{\Delta} := {}^{\mathrm{wSL}_n}\!/_{\Delta}$ the quotient of wSL_n under Δ–equivalence.

In the context of usual links, the *crossing change* is an elementary unlinking operation, that switches a positive classical crossing to a negative one, or vice-versa. In the welded (and virtual) settings, the crossing change is no longer an unlinking operation, but the *virtualization*, defined as the operation that switches a classical crossing to a virtual one, or vice-versa, is a natural generalization which does unlink any welded link. In this paper, we shall consider only the "self"-restriction of these moves. They have been introduced and studied in [7] (for the classical crossing change) and [1] (for the virtualization).

Definition 1.9. An SC move is a crossing change involving two portions of the same strand. We call SC–equivalence the equivalence relation on welded string links generated by SC–moves. We denote by $\mathrm{wSL}_n^{\mathrm{SC}} := {}^{\mathrm{wSL}_n}\!/_{\mathrm{SC}}$ the quotient of wSL_n under SC–equivalence.

Definition 1.10. An SV move is a virtualization move involving two portions of the same strand. We call SV–equivalence the equivalence relation on welded string links generated by SV–moves. We denote by $\mathrm{wSL}_n^{\mathrm{SV}} := {}^{\mathrm{wSL}_n}\!/_{\mathrm{SV}}$ the quotient of wSL_n under SV–equivalence.

Since a crossing change can be realized as a sequence of two (de)virtualization moves, the SC–equivalence is clearly sharper than the SV–equivalence.

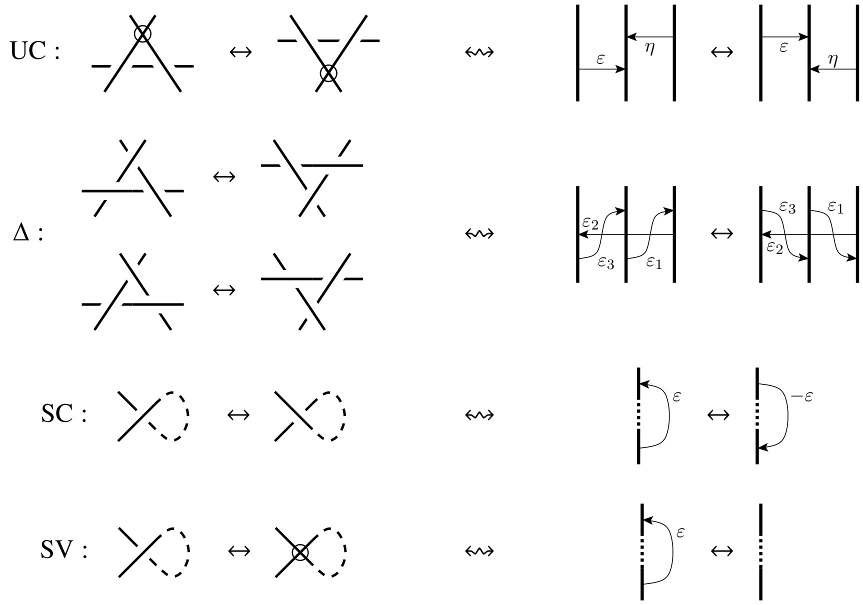


FIGURE 3. Local moves on virtual and Gauss diagrams

All the local moves and equivalence relation above can also be defined for welded links, and we define similarly $w\mathcal{L}_n^{\mathrm{F}}$, $w\mathcal{L}_n^{\Delta}$, $w\mathcal{L}_n^{\mathrm{SC}}$ and $w\mathcal{L}_n^{\mathrm{SV}}$.

It is easily checked that virtual linking numbers descend to invariants for each of the quotients of wSL_n and $w\mathcal{L}_n$ defined in this section.

1.2. **The classical case.** Let us first recall that the (usual) string link monoid SL_n, introduced by Habegger and Lin in [7], corresponds to the set of n-component string link diagrams with no virtual crossing, up to only classical Reidemeister moves. In the same way, classical n–component links correspond to the set $\mathcal{L}_n$ of n–component link diagrams with no virtual crossing, up to only classical Reidemeister moves.

As briefly mentioned in the introduction, the question of whether the natural maps from $\mathcal{L}_n$ to $w\mathcal{L}_n$ and from SL_n to wSL_n are injective remains open; we will denote both maps by $u_{\to}w$. By abuse of notation and according to the context, SL_n will denote both SL_n and $u_{\to}w(\mathrm{SL}_n)$ and $\mathcal{L}_n$ will denote both $\mathcal{L}_n$ and $u_{\to}w(\mathcal{L}_n)$. In the following we will also denote string links up to the SC move by $\mathrm{SL}_n^{\mathrm{SC}}$, string links up to the SV move by $\mathrm{SL}_n^{\mathrm{SV}}$, string links up to Δ moves by SL_n^{Δ}, and finally string links up to fused isotopy (i.e. up to both forbidden moves) by $\mathrm{SL}_n^{\mathrm{F}}$.

Notice that Δ and SC moves are "genuine" moves in the usual realm of (string) links, whereas F and SV moves concern string links considered as welded string links, *i.e.* as elements of $u_{\to}w(\mathrm{SL}_n)$. Let us also recall that the classical notion of link-homotopy is the equivalence relation on SL_n generated by self-crossing changes; it was introduced for links by Milnor in [14], and later used by Habegger and Lin for string links [7].

The following theorems summarize known results on classical (string) links up to F, Δ, SC or SV–equivalence.

Theorem 1.11. [15, 5] *Let L_1 and L_2 be two n–component links. The following assertions are equivalent:*

(1) *L_1 and L_2 are F–equivalent;*
(2) *L_1 and L_2 are Δ–equivalent;*
(3) *$lk_{i,j}(L_1) = lk_{i,j}(L_2)$ for all $1 \leq i < j \leq n$.*

Proof. The proof is just a composition of two independent results: Murakami and Nakanishi proved in [15] that two n–component links L_1 and L_2 are Δ–equivalent if and only if $\mathrm{lk}_{i,j}(L_1) = \mathrm{lk}_{i,j}(L_2)$ for all $1 \leq i < j \leq n$;

Fish and Keyman proved in [5] (see also [3] for a shorter proof) that linking numbers characterize links up to F–equivalence. □

The next result addresses SC- and SV–equivalence for string links.

Theorem 1.12. [2] *Let Λ_1 and Λ_2 be two n–component classical string links. The following assertions are equivalent:*

(1) *Λ_1 and Λ_2 are SC–equivalent;*
(2) *Λ_1 and Λ_2 are SV–equivalent.*

Let us now explore the relation between the SC and SV–equivalence on the one hand, and Δ moves and fused isotopy on the other hand. In [7], Habegger and Lin classified string links up to SC–equivalence. Before stating their result, let us recall some notation. We denote by RF_n the reduced free group of rank n, defined as the quotient group of the free group on n generators $x_1, \dots, x_n$ by the normal closure of the subgroup generated by elements $[x_i, g^{-1}x_i g]$ ($i \in \{1, \cdots, n\}$ and $g \in \mathrm{F}_n$). We define

- $\mathrm{Aut}_{\mathrm{C}}(\mathrm{RF}_n) := \{f \in \mathrm{Aut}(\mathrm{RF}_n) \,\big|\, \forall i \in \{1, \cdots n\}, \exists g \in \mathrm{RF}_n, f(x_i) = g^{-1}x_i g\}$;
- $\mathrm{Aut}^0_{\mathrm{C}}(\mathrm{RF}_n) := \{f \in \mathrm{Aut}_{\mathrm{C}}(\mathrm{RF}_n) \,\big|\, f(x_1 \cdots x_n) = x_1 \cdots x_n\}$.

Habegger-Lin's result can then be stated as follows.

Theorem 1.13. [7] *The monoid SL_n^{SC} is a group, and is isomorphic to the group $Aut^0_C(RF_n)$.*

We can therefore gather and reformulate the above results as follows.

Proposition 1.14. *Let $n \geq 2$.*

(1) *The groups SL_n^{SC} and SL_n^{SV} are isomorphic to $Aut^0_C(RF_n)$;*
(2) *The groups SL_n^{Δ} and SL_n^{F} are isomorphic to $\mathbb{Z}^{\frac{n(n-1)}{2}}$.*

Note that the Δ move is an unknotting operation for SL_1 [15], as is of course the self-crossing change. Moreover, $\mathrm{SL}_2^{\mathrm{SC}}$ is isomorphic to $\mathbb{Z}$ (see the remark following [2, Question 5.5]), and $\mathrm{Aut}^0_{\mathrm{C}}(RF_n)$ is not abelian for $n > 2$. It follows that $\mathrm{SL}_n^{\mathrm{SC}}$ and SL_n^{Δ} coincide only for $n = 1, 2$.

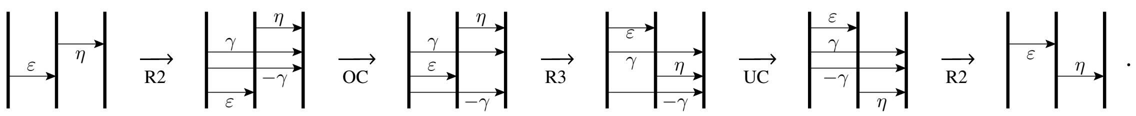
2. The welded case

This section contains all the main results of this paper.

2.1. **Fused string links.** We begin with a lemma on Gauss diagrams.

Lemma 2.1. *Every commutation of arrows on a Gauss diagram can be replaced by a sequence of OC, UC and Reidemeister moves.*

Proof. A commutation of two tails or two heads is just an occurence of an OC or a UC move, respectively. The commutation of an head with a tail can be replaced by the following sequence of moves :



Note that the restrictions on signs requested to perform the R3 move can be fulfilled, since we are free to choose the value of γ and free to choose the orientation of the piece of strand that support the tail of the ε–labeled arrow. □

Remark 2.2. In contrats to the above result, the pure unrestricted virtual braid group on n strands, introduced in [3] and which can be defined as the group of Gauss diagrams with horizontal arrows on n vertical strands up to OC, UC and Reidemeister moves, is actually isomorphic to the cartesian product of $n(n-1)/2$ copies of F_2 (see [3]); in other words, there are arrows which do not commute. This difference lies in the fact that in Lemma 2.1 we allowed for the introduction of non horizontal arrows and, in particular, self-arrows. Indeed, in the figure proving Lemma 2.1, the first and third strands may be part of the same component if the two initial arrows have endpoints on the same two strands. In this case, the R2 move creates two self-arrows.

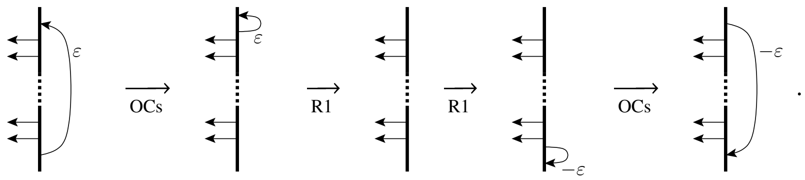


FIGURE 4. Simulating a self-crossing change when there is no head obstruction

Corollary 2.3. *Any two SV–equivalent welded string links are F–equivalent.*

Proof. An SV move corresponds to the removal or addition of a self-arrow. It is therefore enough to prove that we can remove/add one self-arrow using only forbidden moves. But this is a straightforward consequence of Lemma 2.1, and of the fact that we can add or remove an isolated self-arrow using a R1 move. □

A noteworthy consequence of Lemma 2.1 is that, up to F–equivalence, the notions of string links and links coincide. This should be compared with the classical case where the closure map induces a one-to-one correspondence between SL_n and $\mathcal{L}_n$ only for $n = 1$, and to the welded case where it is not even true for $n = 1$.

Corollary 2.4. *For every $n \in \mathbb{N}^*$, the closure map induces a one-to-one correspondence between wSL_n^F and $w\mathcal{L}_n^F$.*

Proof. The operation which closes a string link into a link is always well defined but, in general, it has no inverse since cutting a link into a string link depends on where the cut is performed. However, up to F–equivalence, Lemma 2.1 implies that an arrow endpoint can be freely moved from one endpoint of a strand to the other. This provides a well defined inverse for the closing operation. □

Lemma 2.1 also implies a classification result for fused objects.

Theorem 2.5. *Two welded (string) links Λ_1 and Λ_2 are F–equivalent if and only if, for all $i \neq j \in [\![1, n]\!]$, $vlk_{i,j}(\Lambda_1) = vlk_{i,j}(\Lambda_2)$. In other words, fused (string) links are classified by the virtual linking numbers.*

Proof. For every $k \in \mathbb{N}$, $\varepsilon \in \{\pm 1\}$ and $i_0 \neq j_0 \in \{1, \dots, n\}$, the Gauss diagram $G^{\varepsilon k}_{i_0,j_0}$ which has only k horizontal ε–labeled arrows from strand i to j satisfies $\mathrm{vlk}_{i_0,j_0}(G^{\varepsilon k}_{i_0,j_0}) = \varepsilon k$ and $\mathrm{vlk}_{i,j}(G^{\varepsilon k}_{i_0,j_0}) = 0$ for $(i, j) \neq (i_0, j_0)$. By stacking such Gauss diagrams in lexicographical order of $i \neq j \in \{1, \dots, n\}$, we obtain normal forms realizing any configuration of the virtual linking numbers.

Now, given any Gauss diagram, one can use Corollary 2.3 to remove all self-arrows up to F–equivalence, and Lemma 2.1 to reorganize arrow ends in order to obtain one of the above normal forms. □

Corollary 2.6. *For $n = 2$, $wSL_2^F = wSL_2^{SV}$ and for $n \geq 3$, wSL_n^F is a proper quotient of wSL_n^{SV}.*

Proof. Fused string links are classified by virtual linking numbers, i.e. they are in one to one correspondence with $\mathbb{Z}^{n(n-1)}$. On the other hand, $\mathrm{wSL}_n^{\mathrm{SV}}$ is isomorphic to $\mathrm{Aut_C}(RF_n)$ [1]. The statement therefore follows from the fact that $\mathrm{Aut_C}(RF_2)$ is isomorphic to $\mathbb{Z}^2$ (see proof of Lemma 4.12 of [2]) while $\mathrm{Aut_C}(RF_n)$ is not abelian for $n > 2$. □

2.2. **Unknotting properties of the Δ move.** The Δ move is known to be an unknotting operation for usual knots. We show below that this remains true for welded knots, but not for links with higher number of components. This follows from the following (more technical) result.

Theorem 2.7. *Any two SC–equivalent welded string links are Δ–equivalent.*

Proof. We shall adopt the Gauss diagram point of view. The statement is proven by simulating a crossing change on a self-arrow a, using only Reidemeister, OC and Δ moves. We shall proceed by induction on the *width* of a, which is defined as the number of heads located on the portion of strand in-between the two endpoints of a.

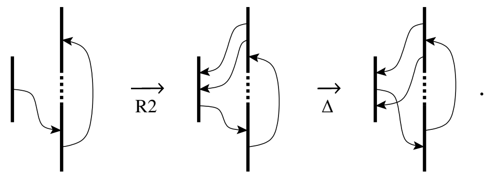


FIGURE 5. Crossing a head of a non interior arrow

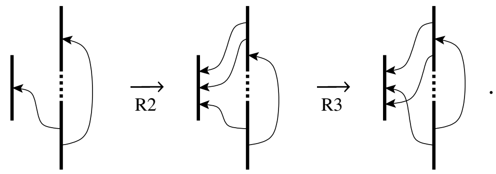


FIGURE 6. Crossing a tail of a non interior arrow

If a has width zero, then there is no head, and using OC moves, the tail of a can be freely pushed nearby its head, so a can be removed using R1. Using R1 again, another arrow with opposite sign can be created at the initial position of the tail of a, and the tail of this new arrow can be pushed, using OC, to the initial position of the head of a. See Figure 4 for an illustration.

Now, we assume that a has width $d \in \mathbb{N}^*$, and that the statement is proven for self-arrows having width $< d$. We call an *interior* arrow any self-arrow which has both endpoints located in the portion of strand between the endpoints of a. There are hence two cases:

(1) There is an interior arrow b; then we proceed in three steps.
  - **Step 1:** Remove b by pushing its tail next to its head, as follows. Tails can be crossed using OC moves. Heads from non interior arrows can be crossed using the sequence of moves described in Figure 5 (note that the restrictions on signs requested to perform the Δ move can be fulfilled, since we are free to choose the signs of the arrows created with the R2 move and free to choose the orientation of the piece of strand that supports the tail of the non interior arrow). Heads from interior arrows can be crossed by using the induction hypothesis, which allows to turn them into tails using self-crossing changes; an interior arrow has indeed a strictly smaller width than a.
  - **Step 2:** The arrow b can now be removed using a R1 move. Since none of the operations of Step 1 has increased the number of head between its endpoints, a has now width $d - 1$ and the induction hypothesis can be used to perform a self-crossing change on it.
  - **Step 3:** The arrow b can be replaced back by performing Step 1 backwards.

(2) There is no interior arrow; then we also proceed in three steps.
  - **Step 1:** Push the tail of a towards its head until it has crossed one head. In doing so, the tail of a first crosses a number of tails (of non interior arrows), and we request that these are not crossed using OC moves, but using the sequence of moves[1] described in Figure 6. Finally, the first head met by the tail of a is crossed using the sequence of move described in Figure 5.
  - **Step 2:** Since none of the operations of Step 1 has increased the number of head between its endpoints, a has now width $d - 1$ and the induction hypothesis can be used to perform a self-crossing change on it.
  - **Step 3:** The tail of a can now be pushed back to its initial position by performing Step 1 backwards. It is indeed illustrated in Figure 7 that a Δ move as in Figure 5 (resp. an R3 move as in Figure 6) performed in Step 1, and the corresponding move performed in this final step, have

[1] The restrictions on signs requested to perform the R3 move can be satisfied for the same reasons than for the Δ move in Figure 5, see Step 1 of the previous case.

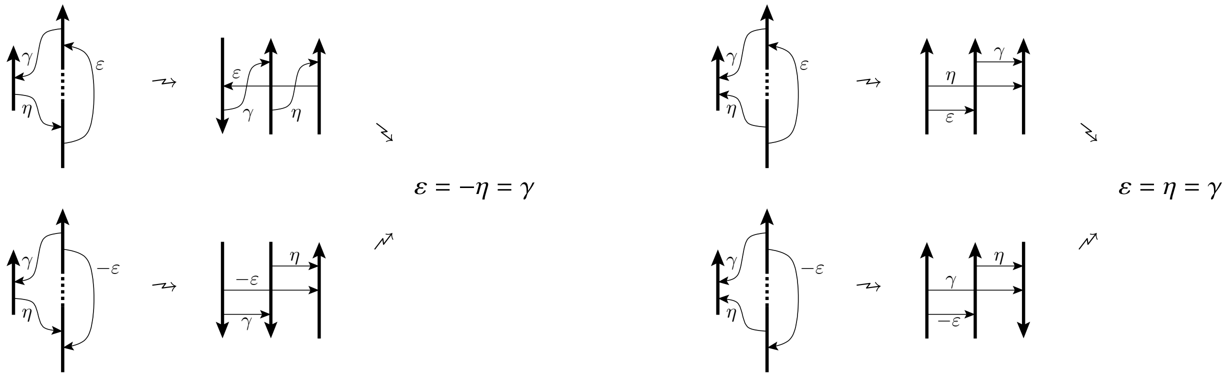


FIGURE 7. Correspondence between signs restrictions

signs restrictions which are simultaneously satisfied. Some random orientations have been chosen for the strands in Figure 7, but changing it would merely add a sign on both sides.

□

Corollary 2.8. *The Δ move is an unknotting operation for long welded knots, hence for welded knots.*

Proof. It was proven in [2] that the SC–equivalence is an unknotting operation for long welded knots. The statement therefore follows from Theorem 2.7. □

Corollary 2.9. *For $n = 2$, we have $wSL_2^{\Delta} = wSL_2^{SC}$, and for $n \geq 3$, wSL_n^{Δ} is a proper quotient of wSL_n^{SC}.*

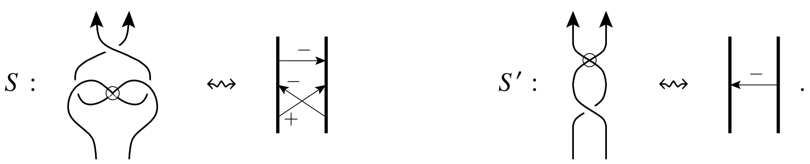
Proof. For $n = 2$, it is conversely true that Δ–equivalence implies SC–equivalence. Indeed, in any Δ move, at least two of the three involved pieces of strand belong to the same strand. By performing a self-crossing change on the corresponding crossing before and after, any Δ move can hence be replaced by an R3 move.

By contrast, for $n \geq 3$, the Milnor invariant $\mu^{w}_{i_1 i_2 i_3}$, defined in [1, Sec. 5.2] but which can also be described in terms of Gauss diagram formula as $\langle \ + \ - \ , - \rangle$ (using notation from [2, Sec. 3.2]), detects the Δ move, but is an invariant of SC–equivalence. □

2.3. **F versus Δ.** It was noted in [2] that the SV–equivalence is a strictly stronger notion than the SC–equivalence for welded (string) links, and that both collapse to the same notion on usual (string) links seen as a subset of welded objects. In that sense, F–equivalence is to Δ–equivalence what SV–equivalence is to SC–equivalence. This analogy is also supported by the various classification results for these equivalence relations, as outlined in the introduction.

Proposition 2.10. *Any two Δ–equivalent welded string links are F–equivalent; but if $n \geq 2$, there are F–equivalent string links which are not Δ–equivalent.*

Proof. The first statement is a direct consequence of Lemma 2.1. For the second statement, we consider the following string links:



Clearly, S and S' are F–equivalent; but they are not Δ–equivalent. Indeed, consider the invariant[2] Q_2, defined as $\langle \ - \ + \ - \ + \ - \ , - \rangle$ in [2], which is an invariant for welded string links up to SC

[2]For $n \geq 3$, the invariant Q_2 requires the choice of two strands

moves. It is also an invariant of Δ–equivalence. Indeed, if a Δ move involves three distinct strands, only one of the three arrows affected by this move can be involved in the computation of Q_2, so that the value of Q_2 is the same before and after the move; if a Δ move involves only one or two distinct strands, then it was noticed in the proof of Corollary 2.9 that it can be replaced by one R3 and two SC moves. The invariant Q_2 is hence well defined on $\mathrm{wSL}_n^{\mathrm{F}}$, and it is directly computed that $Q_2(S) = -1$ and $Q_2(S') = 0$. □

We can now state the analogue of Theorem 1.11 in the context of classical string links.

Proposition 2.11. *Let $n \geq 2$, and let Λ_1, Λ_2 be two n–component classical string links. The following statements are equivalent:*

1. *Λ_1, Λ_2 are F–equivalent;*
2. *Λ_1, Λ_2 are Δ–equivalent;*
3. *$lk_{i,j}(\Lambda_1) = lk_{i,j}(\Lambda_2)$ for all $1 \leq i < j \leq n$.*

Proof. If Λ is a fused string link with only classical crossings, then $\mathrm{vlk}_{i,j}(\Lambda) = \mathrm{vlk}_{j,i}(\Lambda) = \mathrm{lk}_{i,j}(\Lambda)$. So, if two welded string links with only classical crossings Λ_1 and Λ_2 are F–equivalent, then they have the same virtual linking numbers by Theorem 2.5, and hence the same classical linking numbers. But it is proven in [13, Section 4] that if two classical string links have the same classical linking numbers, then they are Δ–equivalent. Finally, by Proposition 2.10, two Δ–equivalent string links are F–equivalent. □

We state now three direct consequences of Proposition 2.11 on classical objects.

Corollary 2.12. *For every $n \in \mathbb{N}^*$, the closure map induces a one-to-one correspondence between SL_n^Δ and $\mathcal{L}_n^\Delta$.*

Proof. This is a consequence of Proposition 2.11 and Corollary 2.4. Note that it can also be seen as a corollary of [15, Theorem 1.1] and its string link counterpart given in [13, Section 4]. □

Corollary 2.13. *A welded (string) link L is F–equivalent to a classical (string) link if and only if, for every $1 \leq i < j \leq n$, we have $vlk_{i,j}(L) = vlk_{j,i}(L)$.*

In Proposition 2.11, we were considering SL_n up to Δ moves; the following corollary shows that it is the same as considering $u_{\to}w(\mathrm{SL}_n)$ up to Δ move.

Corollary 2.14. *(String) links up to Δ moves embed in welded (string) links up to Δ moves.*

Proof. If two welded string links with only classical crossings are Δ–equivalent, they have the same linking numbers; it follows therefore from Proposition 2.11 that they are equivalent as string links up to Δ moves. □

References

[1] B. Audoux, P. Bellingeri, J.-B. Meilhan, and E. Wagner. Homotopy classification of ribbon tubes and welded string links. *ArXiv e-prints*, 2014.

[2] B. Audoux, P. Bellingeri, J.-B. Meilhan, and E. Wagner. On usual, virtual and welded knotted objects up to homotopy. *ArXiv e-prints*, 2015.

[3] V. G. Bardakov, P. Bellingeri, and C. Damiani. Unrestricted virtual braids, fused links and other quotients of virtual braid groups. *to appear in J. Knot Theory Ramifications*, 2015.

[4] Thomas Fiedler. *Gauss diagram invariants for knots and links*, volume 532 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2001.

[5] A. Fish and E. Keyman. Classifying links under fused isotopy. *ArXiv Mathematics e-prints*, 2006.

[6] M. Goussarov, M. Polyak, and O. Viro. Finite-type invariants of classical and virtual knots. *Topology*, 39(5):1045–1068, 2000.

[7] N. Habegger and X.-S. Lin. The classification of links up to link-homotopy. *J. Amer. Math. Soc.*, 3:389–419, 1990.

[8] S. Kamada. Braid presentation of virtual knots and welded knots. *Osaka J. Math.*, 44(2):441–458, 2007.

[9] T. Kanenobu. Forbidden moves unknot a virtual knot. *J. Knot Theory Ramifications*, 10(1):89–96, 2001.

[10] L. H. Kauffman. Virtual knot theory. *European J. Combin.*, 20(7):663–690, 1999.

[11] L. H. Kauffman. A survey of virtual knot theory. In *Knots in Hellas '98 (Delphi)*, volume 24 of *Ser. Knots Everything*, pages 143–202. World Sci. Publ., River Edge, NJ, 2000.

[12] L. H. Kauffman and S. Lambropoulou. Virtual braids and the *L*-move. *J. Knot Theory Ramifications*, 15(6):773–811, 2006.

[13] J.-B. Meilhan. On Vassiliev invariants of order two for string links. *J. Knot Theory Ram.*, 14(5):665–687, 2005.

[14] John Milnor. Link groups. *Ann. of Math. (2)*, 59:177–195, 1954.

[15] H. Murakami and Y. Nakanishi. On a certain move generating link-homology. *Math. Ann.*, 284(1):75–89, 1989.

[16] S. Nelson. Unknotting virtual knots with Gauss diagram forbidden moves. *J. Knot Theory Ramifications*, 10(6):931–935, 2001.

[17] S. Satoh. Crossing changes, delta moves and sharp moves on welded knots. *ArXiv e-prints*, 2015.

[18] V. V. Vershinin. On homology of virtual braids and Burau representation. *J. Knot Theory Ramifications*, 10(5):795–812, 2001. Knots in Hellas '98, Vol. 3 (Delphi).

Aix Marseille Université, I2M, UMR 7373, 13453 Marseille, France
E-mail address: benjamin.audoux@univ-amu.fr

Université de Caen, LMNO, 14032 Caen, France
E-mail address: paolo.bellingeri@unicaen.fr

Université Grenoble Alpes, IF, 38000 Grenoble, France
E-mail address: jean-baptiste.meilhan@ujf-grenoble.fr

IMB UMR5584, CNRS, Univ. Bourgogne Franche-Comté, F-21000 Dijon, France
E-mail address: emmanuel.wagner@u-bourgogne.fr